MATH 1010E Week 11 Lecture Notes (Martin Li)

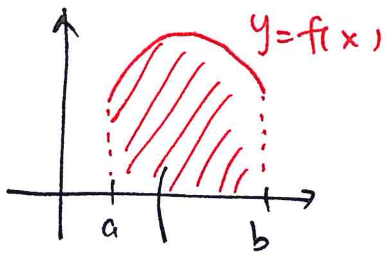
Last time... Integration(s)!

F(x) is primitive function.

① **Indefinite Integrals:** If $F'(x) = f(x)$, then

$$\int f(x)\,dx = F(x) + C$$

② **Definite Integrals:** Area under graph $y = f(x)$.



Area $= \int_a^b f(x)\,dx$

If f is cts, then

$$\int_a^b f(x)\,dx = \underbrace{\lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\,\Delta x_k}_{\text{Riemann sum.}}.$$

③ **Fundamental Thm of Calculus:**

If $F'(x) = f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$

i.e. Definite integrals can be calculated by

(1) doing indefinite integral

(2) then substitute.

Main Question: Given $f(x)$, when can we find $F(x)$?.

- Not trivial. e.g. $\int x^2 \sin(x^x)\,dx = ?$ Q: Does $F(x)$ exist?

Fundamental Theorem of Calculus I

If $f : [a,b] \to \mathbb{R}$ is continuous,

then the function $F : [a,b] \to \mathbb{R}$ defined by

$$\boxed{F(x) := \int_a^x f(t)\,dt}$$

is well-defined and F is differentiable in (a,b)

$$F'(x) = f(x) \qquad \forall x \in (a,b).$$

Remark: (1) f cts $\Longrightarrow$ F exists (answers Main Question)

Note: Even when f is not cts, then $\int_a^x f(t)\,dt$ may still exist.

(2) Only existence result. This does not help you find $F(x)$ explicitly (in formulas).

Preliminary Results about Definite Integrals

(I) Integral Mean Value Theorem:

Let $f : [a,b] \to \mathbb{R}$ be continuous.

$$\int_a^b f(x)\,dx = f(\xi)(b-a) \quad \text{for some } \xi \in (a,b).$$

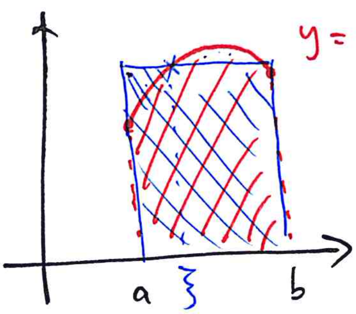
Diff. MVT

$$\frac{f(b)-f(a)}{b-a} = f'(\xi)$$

$\xi \in (a,b)$

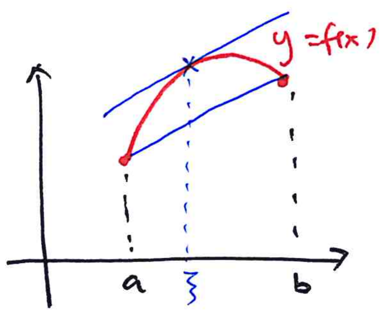
Pictures:

Integration:

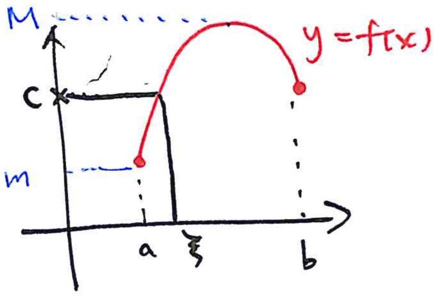


Blue area = Red Area

Differentiation:



The proof requires 2 ingredients :



(i) <u>Intermediate Value Theorem</u>

If $f : [a,b] \to \mathbb{R}$ is cts ,

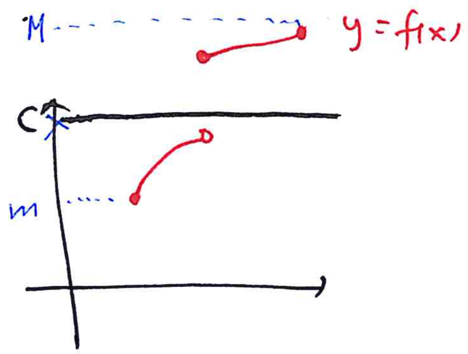
and let $m = \min f$ & $M = \max f$ on $[a,b]$.

then $\forall\, c \in [m, M]$, $\exists\, \xi \in [a,b]$ st.

$$f(\xi) = c \, .$$

<u>Remark</u>: (a) ξ is <u>NOT</u> unique

(b) cts is required :



(ii) <u>Sign Preservation</u>

If $g(x) \leq f(x) \leq h(x)$ on $[a,b]$, all cts functions

then $\boxed{\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b h(x)}$

<u>Ex</u>: (1) Reduce this to prove : $f(x) \geq 0 \Rightarrow \int_a^b f(x)\,dx \geq 0$.

(2) Do we have something similar for differentiation ?

ie $f(x) \geq 0 \overset{?}{\Rightarrow} f'(x) \geq 0$.

<u>Proof of Integral Mean Value Theorem</u>

Let $f : \underline{[a,b]} \to \mathbb{R}$ be $\underline{\text{cts}}$. Then ,

$m = \min f$ and $M = \max f$ exist (Extreme Value Thm)

ie $m \leq f(x) \leq M \quad \forall\, x \in [a,b]$

$$\Rightarrow \quad \frac{m}{b-a} \leq \frac{f(x)}{b-a} \leq \frac{M}{b-a} \qquad \forall\, x \in [a,b]$$

Integrate over $[a,b]$, using (ii),

$$\int_a^b \frac{m}{\underline{b-a}}\,dx \le \int_a^b \frac{f(x)}{b-a}\,dx \le \int_a^b \frac{M}{\underline{b-a}}\,dx$$

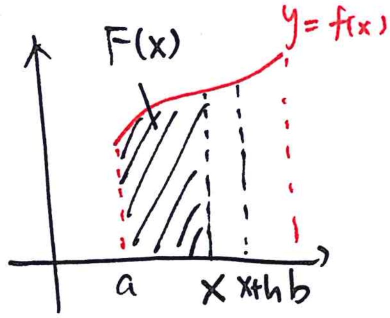
$$\Rightarrow \quad \frac{m}{\cancel{b-a}}\cdot\cancel{(b-a)} \le \underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{c} \le \frac{M}{\cancel{b-a}}\cdot\cancel{(b-a)}$$

Intermediate Value Thm $\Rightarrow$

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(\xi) \quad \text{for some } \xi \in [a,b]$$

$$\Rightarrow \quad \int_a^b f(x)\,dx = f(\xi)\cdot(b-a) \qquad \blacksquare$$

Now, integral MVT proved.



Proof of Fundamental Theorem of Calculus I

Given f cts on $[a,b]$, so

$$F(x) := \int_a^x f(t)\,dt$$ is well-defined since f is cts on $[a,x]$

Check: $F'(x) = f(x) \quad \forall x \in (a,b)$

Back to definition,

$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right]$$

$$= \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt\right]$$

$$= \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

$$= \frac{1}{\cancel{h}}\left[f(\xi)\cdot\cancel{h}\right] \quad \text{for some } \xi \in [x, x+h]$$

Take $h \to 0$, $F'(x) = \lim_{\xi \to x} f(\xi) = f(x)$ since f is cts. $\blacksquare$

Recall: $\underbrace{F'(x) = f(x)}_{\Updownarrow}$ where $F(x) = \int_a^x f(t)\,dt$.

ie
$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

$\Rightarrow$ Integrate then differentiate gives back the original function.

Q: What about differentiate first, then integrate?

Fundamental Theorem of Calculus II

(Recall: $\int_a^b F'(x)\,dx = F(b) - F(a)$)

Let $F : [a,b] \to \mathbb{R}$ be a differentiable function and $F'(x)$ is cts on $[a,b]$.

Then,
$$\boxed{\int_a^b F'(x)\,dx = F(b) - F(a)}$$

ie
$$\int_a^b \frac{d}{dx} F(x)\,dx = F(b) - F(a).$$

$\Rightarrow$ Diff. first then integrate gives back "original function".

Picture:

$$\begin{tikzcd}
\text{functions} \arrow[r, bend left, "\frac{d}{dx}"] & \text{functions} \arrow[l, bend left, "\int dx"]
\end{tikzcd}$$

inverse process.

Proof: Define $G(x) := \int_a^x f(t)\,dt$. $\overset{\text{Fund. Thm. I}}{\Longrightarrow}$ $G'(x) = f(x)$.

If we let $H(x) := F(x) - G(x)$

$\Rightarrow$ $H'(x) = F'(x) - G'(x) = f(x) - f(x) = 0 \quad \forall x \in (a,b)$

$\Rightarrow$ $H(x) \equiv \text{constant}$ on $[a,b]$

In particular, $H(a) = H(b)$.

$$H(a) := F(a) - G(a) = F(a) - \int_a^a \cancel{f(t)\,dt}^{\,0} = F(a).$$

$$\| $$

$$H(b) := F(b) - G(b) = F(b) - \int_a^b f(t)\,dt$$

$$\int_a^b F'(x)\,dx = \int_a^b f(t)\,dt = F(b) - F(a)$$

From now on, focus on Calculations of Integrals.

Q: How to find indefinite integrals? ($\Rightarrow$ definite integrals can then be found by substitutions!)

- elementary functions eg $\int x^n\,dx$, $\int \sin x\,dx$ or $\int e^x\,dx$ ...
- linearity: $\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
- u-substitution: (Chain Rule)

eg $\int \frac{1}{x+1}\,dx = \int \frac{1}{x+1}\,d(x+1) = \ln|x+1| + C.$

Integration by Part

Recall: (Product Rule) $\quad (uv)' = u'v + uv'$

ie $\quad d(uv) = v\,du + u\,dv$

Fund. Thm.

$$\Rightarrow \quad uv = \int d(uv) = \int v\,du + \int u\,dv$$

$$\Rightarrow \quad \boxed{\int v\,du = uv - \int u\,dv}$$

switch u & v $\Rightarrow$ extra minus sign

$\Rightarrow$ extra term $= uv$.

Examples

(1) $$\int \underbrace{\ln x}_{v}\, d\underbrace{x}_{u} = \underbrace{x\ln x}_{uv} - \int \underbrace{x}_{u}\, d\underbrace{(\ln x)}_{v}$$

$$= x\ln x - \int \cancel{x}\cdot\left(\frac{1}{\cancel{x}}dx\right)$$

$$= x\ln x - x + C \quad ※.$$

Check: $(x\ln x - x)' = x\cdot\frac{1}{x} + \ln x - 1 = \ln x.$

(2) $$\int x^2 e^{-2x}dx = \int \frac{x^2}{-2} d(e^{-2x})$$

$$= -\int e^{-2x} d\left(\frac{x^2}{-2}\right) + \frac{x^2e^{-2x}}{-2}$$

$$= \int xe^{-2x}dx - \frac{1}{2}x^2e^{-2x}$$

$$= \int \frac{x}{-2} d(e^{-2x}) - \frac{1}{2}x^2e^{-2x}$$

$$= -\int e^{-2x} d\left(\frac{x}{-2}\right) - \frac{1}{2}xe^{-2x} - \frac{1}{2}x^2e^{-2x}$$

$$= \frac{1}{2}\int e^{-2x}dx - \frac{1}{2}xe^{-2x} - \frac{1}{2}x^2e^{-2x}$$

$$= \frac{-1}{4}\int e^{-2x}d(-2x) - \frac{1}{2}xe^{-2x} - \frac{1}{2}x^2e^{-2x}$$

$$= -\frac{1}{4}e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{2}x^2e^{-2x} + C$$

$$= -\frac{1}{4}e^{-2x}(1+2x+2x^2) + C$$

※.

(3) $\int x \tan^{-1}x \, dx = \int \tan^{-1}x \, d\left(\frac{x^2}{2}\right)$

$$= \frac{x^2}{2}\tan^{-1}x - \int \frac{x^2}{2} \, d(\tan^{-1}x)$$

$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$$= \frac{x^2}{2}\tan^{-1}x - \int \frac{x^2}{2} \, \frac{1}{1+x^2} \, dx$$

$\frac{d}{dx}(x\tan^{-1}x)$

$= \tan^{-1}x + \frac{x}{1+x^2}$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \underbrace{\frac{(1+x^2)-1}{1+x^2}}_{1-\frac{1}{1+x^2}} \, dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left[x - \tan^{-1}x\right] + C$$

#.

(4) $\int \sin^{-1}x \, dx = x\sin^{-1}x - \int x \, d(\sin^{-1}x)$

$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

$$= x\sin^{-1}x - \int x \cdot \frac{1}{\sqrt{1-x^2}} \, dx$$

$$= x\sin^{-1}x + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}} \, d(1-x^2)$$

$$= x\sin^{-1}x + \cancel{\frac{1}{2}} \, \frac{\sqrt{1-x^2}}{\cancel{1/2}} + C$$

$$= x\sin^{-1}x + \sqrt{1-x^2} + C$$

#.

Last time ... Fund. Thm of Calculus

$$F'(x) = f(x) \quad \Rightarrow \quad \int_a^b f(x)\,dx = F(b) - F(a).$$

So, focus on indefinite integrals.

- u-substitution
- integration by part $\quad \int u\,dv = -\int v\,du + uv$
- trigonometric functions

e.g.: $\displaystyle \int \sin^2 x \cos^3 x \, dx \quad$ or $\quad \displaystyle \int \frac{1}{\sqrt{1-x^2}}\,dx$

Trigonometric Identities

(I)
$$\begin{cases} \boxed{\cos^2 x + \sin^2 x = 1} \\ 1 + \tan^2 x = \sec^2 x \\ 1 + \cot^2 x = \csc^2 x \end{cases}$$

(II) Sum-to-Product Formula:

$$\begin{cases} \cos(x+y) = \cos x \cos y - \sin x \sin y \\ \sin(x+y) = \sin x \cos y + \sin y \cos x \\ \tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y} \end{cases}$$

Recall: $e^{i\theta} = \cos\theta + i\sin\theta$. (Euler) $\quad i^2 = -1$

$$\begin{array}{c} e^{i(x+y)} = \cos(x+y) + i\sin(x+y) \\ \| \\ e^{ix} \cdot e^{iy} = (\cos x + i \sin \overset{x}{\cancel{y}}) \cdot (\cos y + i \sin y) \end{array}$$

$\Uparrow$ expand, then compare coefficients

$x=y$

(II) $\Rightarrow$ Double angle formula :

$$\begin{cases} \cos 2x = \cos^2 x - \sin^2 x \\ \sin 2x = 2\sin x \cos x \\ \tan 2x = \dfrac{2\tan x}{1-\tan^2 x} \end{cases} \Rightarrow \begin{cases} \sin^2 x = \dfrac{1-\cos 2x}{2} \\ \cos^2 x = \dfrac{1+\cos 2x}{2} \end{cases}$$

(II) $\Rightarrow$ Product-to-Sum Formula :

$$\begin{cases} \cos x \cos y = \dfrac{1}{2}\left(\cos(x+y) + \cos(x-y)\right) \\ \cos x \sin y = \dfrac{1}{2}\left(\sin(x+y) - \sin(x+y)\right) \\ \sin x \sin y = \dfrac{1}{2}\left(\cos(x-y) - \cos(x+y)\right) \end{cases}$$

Ex: Prove all these formula.

Examples :

(1)
$$\int \sin^4 x\,dx = \int (\sin^2 x)^2\,dx$$
$$= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$
$$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx$$
$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx$$
$$= \frac{1}{4}\left[\frac{3}{2}x - \sin 2x + \frac{\sin 4x}{8}\right] + C. \quad \#.$$

Ex: $\int \sin^{2n} x\,dx$ or $\int \cos^{2n} x\,dx$

(2) $$\int \sin 3x \sin 5x \, dx = \int \frac{1}{2}\left[\cos(-2x) - \cos 8x\right] dx$$

$$= \frac{1}{2}\int (\cos 2x - \cos 8x)\, dx$$

$$= \frac{1}{2}\left[\frac{\sin 2x}{2} - \frac{\sin 8x}{8}\right] + C \; \#.$$

Ex: $\int \cos 2x \sin 3x \, dx$.

(3) $$\int \cos x \sin^4 x \, dx = \int \sin^4 x \, d(\sin x)$$

$$= \frac{\sin^5 x}{5} + C \; \#.$$

Ex: $\int \cos^8 x \sin x \, dx$

(4) $$\int \sin^2 x \cos^2 x \, dx = \int \left(\frac{\sin 2x}{2}\right)^2 dx$$

$$\sin^2 x(1-\sin^2 x) = \sin^2 x - \sin^4 x$$

$$= \frac{1}{4}\int \sin^2 2x \, dx$$

$$= \frac{1}{4}\int \frac{1-\cos 4x}{2}\, dx$$

$$= \frac{1}{4}\left[\frac{1}{2}x - \frac{\sin 4x}{8}\right] + C \; \#.$$

(5) $$\int \sec^3 x \, dx = \int \sec x \, d(\tan x)$$

$$= \sec x \tan x - \int \tan x \, d(\sec x)$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$= \sec x \tan x - \int \underbrace{\tan^2 x}_{\sec^2 x - 1} \sec x \, dx$$

$$= \sec x \tan x - \int (\sec^3 x - \sec x)\, dx$$

$$\Rightarrow \quad 2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| + C$$

$$\Rightarrow \quad \int \sec^3 x\,dx = \frac{1}{2}\Big[\sec x \tan x + \ln|\sec x + \tan x|\Big] + C \quad \#$$

(6)

$$\int \frac{\sin x}{\cos x + \sin x}\,dx = \int \frac{\tan x}{1+\tan x}\,dx$$

$$= \frac{1}{2}\int \left[1 - \tan\left(\frac{\pi}{4} - x\right)\right] dx$$

$$= \frac{1}{2}\left[x + \ln\left|\sec\left(\frac{\pi}{4} - x\right)\right|\right] + C \quad \#$$

$\tan\frac{\pi}{4} = 1$

$\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

$$\tan\left(\frac{\pi}{4} - x\right) = \frac{1 - \tan x}{1 + \tan x} = \frac{(1+\tan x) - 2\tan x}{1+\tan x}$$

$$= 1 - 2\left(\frac{\tan x}{1+\tan x}\right)$$